Reasoning with Orders of Magnitude

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Scientists have always needed to work with very large numbers. Archimedes, the first great scientist and mathematician, had to invent for himself several kinds of mathematics, including a notation for very large numbers. In a booklet called the *Sand Reckoner*, he introduced his notation and calculated the number of sand grains that could fit into the Solar System. As an ancient Greek, Archimedes did not have our modern decimal numeral system, but today any middle school student can perform similar reckoning feats using powers of ten. This paper will teach you how.

The Power of Adding Another Zero

Every small child learns how to count to ten on his own fingers. A little later, he is told that tens themselves can be counted, and the world of numbers opens up to include one hundred. (Many of us remember the first time we counted proudly to one hundred!) But there is no reason to stop at one hundred, and so we go on to learn about a thousand (ten hundreds), a million (a thousand thousands) and even a billion (a thousand million). Perhaps you have learned numbers beyond, such as a quadrillion (which is, of course, a thousand trillion). If you know your Latin, you can figure out that a "centillion" is the 100-th item in a sequence which starts with "million", "billion", "trillion". (Here's an exercise: How many zeroes are in a centillion, when it is written out in decimal notation? Hint: There are 12 in a trillion.) This system of naming is sufficient for most purposes. For example, the sort of numbers we hear about in the news about the national economy are rarely larger than a hundred billion.

But the world of science picks up where the everyday world leaves off. (That is almost the definition of science, if you like.) In science, we talk about the number of stars in our galaxy, or the mass of our earth, or the speed of light, or the size of an atom. Those unusual measurements require special units and unusual numbers. Like Archimedes of old, modern scientists have devised specialized ways of talking about numbers which are more powerful and convienient than ordinary English can do. As we learn to use these numbers and units, we will feel much more at home in the world of science. Even if we do not all become professional scientists, we will be able to talk the language, and evaluate scientific claims. Such skills are necessary to be effective citizens in today's world.

Review Questions:
- When written out in decimal notation, how many zero digits are in a centillion? (*cent.* = 100)
- How many zeroes are in these decimal numerals: octillion, nonillion, decillion.
- In general, how many zeroes are in an "*N*-illion", where *N* is a Latin number word?

Three Scientific Simplifications: Counting Zeroes, Metric Units, Approximation

The key idea scientists use to manage their numbers is to get rid of unnecessary detail. We can do this in three ways. First, we find a way to do without extra words like "trillion" (though they will come back later in a changed form) and learn to count zeroes instead. Second, we learn to use a simplified system of units, the metric system. Third, we learn to work comfortably with approximate numbers, rather than requiring every last digit of every number.

When we first learned as children to count to one hundred, we did it by taking ten steps of ten. At first this was a one-time trick, for getting to one hundred. Scientific notation requires that we be willing to perform this trick any given number of times. You get to a thousand by taking ten steps of ten steps of ten. Instead of talking about ten million, a scientist will note that there are seven zeroes in that number, and simply report it as the seventh *power of ten*. As you have proabably learned in math, the N-th *power of ten* is obtained by multiplying together N distinct copies of the number ten, which of course is written as a digit "one" followed by N zeroes.

A scientist working with large numbers would quickly get tired writing many digits, and so he (or she, of course) prefers to use *scientific notation*, such as 10^7 for ten million (10,000,000). The little number (called an *exponent*) above the ten says how many zeroes you would need to add to a "one" digit, or (equivalently) how many factors of ten to multiply together to reach the desired number. The scientist thus writes the N-th power of ten as 10^N. A number like twenty-one million might be written as 2.1×10^7 or 21×10^6. (Before you go one, be sure you understand how both numerals come to the same number.) A computer or calculator might write the same number more plainly as "2.1e7", where the letter "e" stands for "exponent", or more specifically "-times-ten-to-the-power-of-". So the previous example or 2.1×10^7 can also be written "2.1e7". (This is handy when you don't have fancy fonts to play with.) In essence, an exponent gives a number of "extra" zeroes to add to a numeral, to inflate it to some required grandness of scale.

Using exponential notation, it is easy to write very large numbers, much bigger ones than even Archimedes knew about. For example, a "google" is the one-hundredth power of ten, or 10^{100}. You could contain about a google of smoke particles in the known universe if you packed it full of them. (Later in this paper you'll see how to find that out.) The speed of light is a more reasonable example. It travels about 3×10^8 meters (or yards) per second.

Review Questions
• How does scientific notation differ from other ways to express numbers?
• Write out the speed of light in scientific notation, decimal notation, and English words.
• What is scientific notation best for? When would it be less convenient?

Two More Simplifying Techniques

Besides counting zeroes using exponents, a second trick scientists use to simplify their measurements is to use a smaller set of units, the metric system. (Be sure you know your metric system fundamentals before you go any farther! Know what the following are: meter, kilometer, centimeter, millimeter, gram, and kilogram.) As you probably know, the chief merit of the metric system is that there is just one main unit for each kind of measurement. Length is measured in meters, mass in grams, time in seconds, and so forth. When larger or smaller units are needed, we simply multiply or divide by a power of ten. Kilometers (a thousand meters) measure distances in a city, while millimeters (a thousandth of a meter) measure small objects. (If you want to look at it another way, kilometers, meters, and millimeters are separate units that convert to each other via powers of ten.) We will say more about this later. Just as we chose ten and its powers as our basis for sizing numbers, instead of a dictionary full of numbers ("hundred", "thousand", "million", ...) we choose a meter and its multiples as a basis for measuring lengths, instead of a dictionary full of units ("yard", "inch", "mile", ...). It is interesting to remember that this scientific system, like the more homely English system, is still based on the human body: Ten fingers, a meter as an arm's length.

A third trick scientists use to simplify their notation is to decide that they do not always need complete accuracy. The speed of light is 299,792 kilometers per second, but for most purposes, including most scientific observations, nobody would care if it were 299,790, or 299,800, or even just 300,000 kilometers per second. Even that last, simplest, most rounded number is just a fraction of a percent different from the most accurately known figure. We say that 300,000 kilometers per second is a *good approximation* to the true speed of light. If you ask them for the speed of light, most physicists will quote you the approximate figure, rather than the exact one. In fact, nobody knows the "true" figure for any physical constant, since that would require an infinitely precise measurement. The most accurately known constants are known to about twenty decimal digits. Although progress in technology improves the accuracy of such measurements over time, the improvements are slow and small.

Review Questions:
• What are the three ways scientists simplify their work with complex measurements?
• What are the main differences between the metric system and the English system of units?
• What are some examples of approximation, both rough and precise?

Working with Orders of Magnitude

There is an even simpler way to round numbers, and that is to remember *only* the power of ten that the number is closest to. Forget the digits altogether! For example, the power of ten which is closest to the speed of light is 100,000 kilometers per second, which is 10^8 meters per second, or "10e8" meters per second. (Be sure you understand these last two ways of writing it, before you go any farther!) It is an important realization that you can round away *all* the digits of a number and still have a useful approximation. This special kind of approximation is called an *order of magnitude*. If you learn the sizes and scales of scientific objects just to their orders of magnitude, you will know enough to be very literate in scientific discussions.

If you are given a measurement, it is easy to decide its corresponding order of magnitude. First, throw away all digits except the first, rounding up if the second digit is 5 or greater. Then, if the lone remaining digit is 4, 5, 6, 7, 8, or 9, round it up to 10 (adding another zero). If the digit is 2 or 3, round it down to 1. The result will be a number which is a pure power of ten, and this is the order of magnitude. Oddly enough, the halfway point from which you round is between 3 and 4, not at 5 as one might expect. We will see why this is so later. To give some examples, 300 or 316 round down to 100, but 400 or 350 round up to 1000. A dozen rounds to 10, and there an order of 100 minutes in an hour, 10 hours in a day, and 1000 days in a year!

In order to work with orders of magnitude, it is helpful to learn at least one object which exemplifies each distinct order of magnitude. (All the numbers in this section will be in meters, unless otherwise specified.) The first order of magnitude, ten meters, is about the size of a house, so remember that a house is 10^1. (You can pronounce this as "ten to the first", or simply "the first power".) Whether it is a shack or a mansion, if it is bigger than 3 meters and smaller than 35 meters, it counts as 10^1. Likewise, a sports field, whether for football (100 yards long) or baseball (with 400 feet to center field), counts as as 10^2. (The Great Pyramid of Giza is that order of magnitude also.) A neighborhood of a town is something like 10^3. This can be 3 kilometers or half a mile; the order of magnitude is the same once you round away all the digits. You can continue increasing orders of magnitude, inflating sizes by ten times in each step. After 16 steps you get to a light year, which is the distance light travels in one year. At 9.5 x 10^{15} meters, it's a jumbo-sized unit favored by astonomers.

Here is a list of objects of all the orders of magnitude between your house and a light year:

1	your house	9	Moon orbit, Sun
2	sports field, Great Pyramid	10	4 days of Earth orbit
3	neighborhood	11	Earth orbit (1 AU)
4	Mt. Everest	12	Jupiter orbit (5 AU), Betelgeuse
5	Bay Area, large city	13	Solar system
6	California, large state	14	heliopause (maximum likely)
7	Earth, Jupiter's Great Red Spot	15	(empty space inside Oort cloud)
8	Jupiter	16	light year, Oort cloud

(Notes: Betelgeuse is a red giant star. The heliopause is the point, currently uncertain, at which the solar wind slows down and merges into the thin interstellar medium. The Oort cloud is a theoretical source of comets.)

If you check the actual sizes, you'll find rounding throughout this table. Although the Earth is about 150 million kilometers (1.50 x 10^{11} meters, or one Astronomical Unit) from the Sun, we round down to an order of magnitude of 10^{11} meters. On the other hand, though Jupiter is about

778 million kilometers (7.78×10^{11} meters) from the Sun, we round up to 10^{12} meters.

This system of orders of magnitude works like a ruler on which everything has a measurement. Unlike other rulers, it encompasses all known objects, from the known universe as a whole down to you and me. (As we will see later, it also goes down all the way to the smallest subatomic particles.) The main use of this gigantic ruler is to organize our impressions of the sizes of objects. Once you know a thing's order of magnitude, it is easy to make rough comparisons with other known objects. For example, Mt. Everest is order 4 while your house is order 1, so Mt. Everest is 3 orders larger than your house. What, exactly, does this mean? It means that the number for the height of Mt. Everest is similar to the number of your house's size, except that three more zeroes are added. Thus, Mt. Everest is roughly one thousand (10^3) stories tall. Actually, it's more like two thousand stories, but that amounts to the same order of magnitude as one thousand stories, after you round it.

You can make a similar comparison between any two items in the table above: The Earth's orbit is 3 orders larger than the Moon's orbit, and a light year is 3 orders larger than the Solar System as a whole. Each of the preceding comparisons is between one thing and another thing roughly a thousand times bigger. If you were to make a table-top model one meter in size, with a sand grain a millimeter (one thousandth of a meter) in size, you could visualize the proportion of the Solar System to a light year, or any other pair of items that differ by three orders of magnitude.

This is the real beauty of reckoning with added zeroes: The comparisons are easy to understand. Though the estimates are rough, they capture a useful sense of the overall scale of our world. Scientists routinely use such estimates when they think about the world.

The known universe appears to go out to about 15 billion light years, so there are ten more orders of size magnitude known to science. Here they are:

17	Crab Nebula (M1)	22	*distance* to Andromeda Galaxy
18	globular star cluster (M3)	23	Local Group (galaxy cluster)
19	*distance* to Orion Nebula (M42)	24	Local Supercluster
20	Magellanic Cloud (either one)	25	Great Wall
21	Milky Way Galaxy (diameter)	26	the known universe

(Notes: Galaxies are organized in clusters; ours is called the Local Group, and includes Andromeda also. Clusters, in turn, are organized in superclusters. We are near the edge of the Local Supercluster, at the center of which is the Virgo Cluster. The Great Wall is the largest known structure in the Universe, a sheet of galaxies half a billion light years wide.)

Review Questions:

• What is the order of magnitude of the following numbers: 1, 2, 3, 4, 6, 9, 12, 20, 29, 35, 40.

• How does the Earth compare in size to Jupiter? To the Sun?
• How many distinct orders of magnitude are there in the visible universe?
• Make your own comparison between two objects in your house, and two astronomical objects.

Mapping the Microworld by Subtracting Zeroes

Just as adding an order of magnitude inflates a measurement by ten times, subtracting an order will shrink a measurement to ten times smaller. One direction multiplies by ten, adding a zero; the other direction divides by ten, removing a zero. For example, a centimeter is two orders of magnitude smaller than a meter, and a millimeter is three orders of magnitude smaller. By using negative numbers, we can define orders of magnitude for tiny things as well as big things. Thus, a centimeter is 10^{-2} (or "10e-2") meters, and a millimeter is 10^{-3} (or "10e-3"). (A plain meter is neither a positive nor a negative order of magnitude, so we can call it the "zeroth" order.) Notice that when we write such numbers as decimal fractions, the negative power may be observed by counting the number of zeroes to the *left* of the first digit: 0.01 (1/100) is the second negative power of ten, 0.001 (1/1000) is the third, and 0.000001 (one millionth) is the sixth.

These negative powers can be compared directly with positive powers. For example, a centimeter (10^{-2}) is five orders smaller than a kilometer (10^{3}), because the difference between three and negative two is five. Five orders is 100,000, so it's clear that there are that many centimeters in a kilometer. More interestingly, the size comparison between a centimeter and a kilometer is that same as the comparison between your house and the state of California, or the Earth and Jupiter's orbit, or a light year and the Milky Way Galaxy, or any other two objects which differ by five orders of magnitude.

As with large objects, it is helpful to learn at one or two examples of a small object for each negative power of ten. Here is a list of objects from a decimeter down to an angstrom, which is the unit used to measure individual atoms:

-1	your hand	-6	bacterium, red light wave (micron)
-2	your finger tip, pebble (1 cm)	-7	smoke particle
-3	pencil tip, sand grain (millimeter)	-8	*thickness* of flagellum
-4	*thickness* of human hair	-9	*thickness* of DNA helix
-5	cloud droplet, human cell	-10	atom (angstrom)

(Notes: A micron is a millionth of a meter. An angstrom is a tenth of a billionth of a meter. A flagellum is a tail-like appendage on some cells, used for propulsion. A DNA helix is the strand-like molecule coiled within every cell nucleus which records genetic information; DNA molecules are very long, often centimeters in length. Radiation such as light moves through space in the form of tiny waves, each different color of the rainbow possessing its own wavelength.)

With this information, you can easily see that your house compared to the Earth's orbit (1 to 11), or the Earth's orbit within the Milky Way (11 to 21), or a light year within the known universe (16 to 26), is comparable to an atom within your body (-10 to zero). All of those comparisons differ by ten orders of magnitude. Also, since a cell (at -5) is midway between a plain meter (at zero) and an atom (at -10), it follows that the atoms in a cell are about the same relative size to it as the cells in your body. (Don't forget that we are dealing in approximations, and that human cells vary among themselves in size by an order of magnitude or more.)

At present, the smallest observable size is that of the electron or quark, 10^{-18} meters. There are not many objects smaller than an atom, so a table giving a the rest of the tiny orders turns out to be mostly blank:

-11	hard X-ray wave	-15	proton, neutron
-12		-16	
-13		-17	
-14	nucleus of lead	-18	electron, quark

(Notes: Normal X-ray are atom-sized. Since an atom's nucleus can contain a few hundred protons and neutrons, it can be up to six times larger than a single proton. Quarks are the tiny constituents of larger particles. Physicists classify quarks and electrons as "point particles".)

It is interesting to see how much empty space goes into every atom. An atom is more than four orders larger than one of its own protons. Therefore, if an atom were the size of a house (10 meters), its nucleus would be smaller than a grain of sand (a millimeter).

Review Questions:
• Make your own comparison between two objects in your house, and two microscopic objects.
• How many orders of magnitude are there between a millimeter and a kilometer?

Scientific Terminology (and How to Remember It)

You are already familiar with metric prefixes like "centi-", "milli-", and "kilo-". When talking about very large or very small numbers, scientists and other technical workers prefer to use these prefixes, instead of phrases like "one hundredth of", "one thousandth of", or "one thousand times". Here are the familiar prefixes for the first three powers of ten, both positive and negative:

1	deka-	-1	deci-
2	hecto-	-2	centi-
3	kilo-	-3	milli-

As with traditional terms like "million", "billion", and "trillion", scientists talk about larger powers of ten in groups of three. (You can look at these special powers as powers of 1000.) Here are all the standard prefixes for the positive powers:

3	kilo-	15	peta- (P for Phifteen)
6	mega- (Megabucks millionaire)	18	exa- (E for Eighteen)
9	giga- (Giant, billionaire Gates)	21	zetta- (Zeven threes is 21)
12	tera- (T for Twelve)	24	yotta- (Yolks of 2 dozen eggs)

(Because the less familiar prefixes may be hard to memorize, there are helper phrase for each prefix which you can use to help jog your memory. Such helpers are called “mnemonics”. Or, it may be easier to memorize these all at once as a chant.)

There is also a corresponding set of prefixes for negative powers:

-3	milli-	-15	femto- (F for Fifteen)
-6	micro- (Microscope required)	-18	atto- (A for Ate-teen)
-9	nano- (N for Nine zeroes)	-21	zepto- (Zeven threes is 21)
-12	pico- (Pack a dozen eggs)	-24	yocto- (Yolks of 2 dozen eggs)

Using these prefixes, you can astonish your friends by mentioning that the Milky Way Galaxsy and the Universe are about one zettameter and 130 yottameters wide, or that you prefer your electrons in the one attometer size. More importantly, you will be able to recognize such numbers when you read about them in scientific or technical writing.

Review Questions:
• What is the prefix for a trillion? A trillionth?
• What is a nanosecond? How long is a megasecond?
• What is gigabyte? A megawatt? (You don’t need to explain what a “byte” or “watt” is.)

Squares, Cubes, and Halfway Powers

In both everyday life and in scientific description, we use units of length as a way to talk about areas and volumes. We might say that a table is one square meter, or that a large bathtup holds a cubic meter of water (which, by the way, is the definition of a metric ton). Things get more complicated, as usual, when the numbers get large or small. Though a one meter wide table is a square meter, it is not the case that a three meter wide table is three square meters: It is nine. Where did the extra six square meters come from? The answer is that when you inflate the size of an object (as measured in meters) you inflate its area twice. Tripling the size of your table triples the triple of its area, giving a blow-up factor of nine rather than three. You can see this easily if you cut a square paper into three equal rows and three equal columns (3 x 3 = 9).

Squaring works with powers of ten also. If I upgrade my house of 10 meters to a 100 meter office building, the width of the house grows by 10 times, but the floor space grows by 100 times. My house does not need to be an exact square (like the table was) in order for this to work. Any shape will do. It even works for curved areas, like a spherical surface. For example, because Jupiter is ten times the Earth in size, Jupiter has 100 times as much surface area.

As with squares, so also with cubes. If I upgrade my one meter bathtub to a ten meter diving pool, the new pool holds 1000 cubic meters of water. If my office building has ten times as many stories as my original house, it has 1000 times the volume to air condition. And Jupiter has 1000 times the volume of Earth. (It does not have 1000 times the mass, since the two planets have different densities.) Even though it is not cubical but round, an aluminum nucleus with 27 protons and neutrons is about three times as wide as a single proton (3 x 3 x 3 = 27).

If we go to multiple powers of ten, the rule is to double the number of zeroes when working with areas and triple them when working with volumes. So if you cover a meter-square tabletop with millimeter grains of sand (10^{-3}), there will be a million (10^6) of them. If you fill a cubic meter bathtup with these grains, there will be a billion (10^9) of them in all.

If your body were a true cubic meter, since cells are about 10^{-5} of a meter in size, there would be 10^{15} cells in your body. Since you are slimmer than a true cube, you weigh less than a tenth of a metric ton, so in fact your body contains an estimated 10^{14} cells. It usually works well to estimate the volume of an object by assuming first that it is like a cube, and then (maybe) making adjustments. For example, the Earth (at 10^4 *kilometers*) contains about a trillion (10^{12}) cubic *kilometers* of material. Since a cubic kilometer contains a billion (10^9) cubic *meters*, the Earth also contains about 10^{21} cubic *meters* of material.

This lets us figure out how many smoke particles could fit in the universe, if we had a vast enough supply of them. If the universe is 10^{26} and a smoke particle is 10^{-7} meters, the difference between them is 33 orders of magnitude. (In other words, 10^{33} smoke particles lined up end-to-end would span the universe, more or less.) Since we're talking about volume, we must triple the exponent, to get 10^{99} smoke particles. Make the particles a little smaller or the universe a little bigger (doubling it is enough), and you have a google of particles neatly packaged.

Just as there are occasions for doubling or tripling powers of ten, sometimes it is helpful to split them. Though this is a complicated mathematical concept, one observation is worth making here. There is a natural "half way" point between two orders of magnitude. Just as 10 is halfway between 1 and 100 (as an order of magnitude) because ten tens is one hundred, and 1000 is halfway between 1 and 1,000,000 because a thousand thousands is a million, so 3.16 is halfway between 1 and 10 because the square of 3.16 is (approximately) 10. In practice what this means is that a number like 30 is just about halfway between 10 and 100, as a power of ten. Just as rounding a number like $1.50 is somewhat erroneous because you could equally round to $1 or $2, rounding 30 to either 10 or 100 introduces a similar amount of error.

Sometimes it is best to think of an order of magnitude as being halfway between two "regular" orders. The Moon is not in any of the above tables because, at 3500 kilometers, it is just about halfway (as an order of magnitude) between the size of the Earth (order 7) and the size of California (order 6). In effect, the Moon is of order 6.5. Likewise, the distance light travels in a second, though a very interesting measurement, is not on the tables above, because it falls halfway between the size of Jupiter (order 8) and the size of the Sun (order 9). In effect, it is of order 8.5, at 300 million meters per second (3×10^8). The very nearest stars (Proxima Centauri, Sirius) are at a distance of order 16.5 (a few light years). And a typical atomic nucleus is about three times the size of a proton, so it is of order halfway above a proton, at -13.5.

Review Questions:
• How many square meters are in a square kilometer?
• About how many square meters are on the Earth's surface (double its order of size).
• How does the rounding of orders of magnitude differ from normal decimal rounding?
• Do you think that "google" is a useful number for physics? (Hint: a mathematician invented it.)

Assignment: Things to Remember

You should memorize the following facts from the previous pages. Once memorized, they will serve you as an alphabet for many kinds of scientific and technical reasoning and discussion.
• Example objects for every order, in all tables. Know that Earth is order 7, and so forth.
• Halfway objects and their halfway orders: Moon (6.5), light-second, nearby stars, nucleus.
• The rule for rounding numbers to orders of magnitude (30 goes to 10,.40 goes to 100, etc.)
• The rule of doubling powers when reckoning area.
• The rule of tripling powers when reckoning volume.
• All metric prefixes, large and small, through yotta- and yocto-.
• The following units: light year (about 10^{16} meter) micron (10^{-6} meter), angstrom (10^{-10} meter)

Your teacher will quiz you on these items. A very good way to learn them is with **flashcards**.

Postscript: What Else is Out There?

We have been mainly looking at the sizes of things, using a meter as our main unit, or simply counting numbers of objects. Similar observations can be made about very large or very small measurements of time, frequency, speed, mass, temperature, and more. The electromagnetic spectrum of radiation (which includes visible light) includes frequencies and wavelengths which span twenty orders of magnitude. Temperatures achieved in laboratories have spanned thirty orders of magnitude. Old Archimedes would have been astonished and delighted.